International Journal of Theoretical Physics, Vol. 37, No. 1, 1998

Numerical Computation of Quantum Capacity[3]

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Received July 4, 1997

Quantum capacity is an important tool to analyse the performance of information transmission for quantum channels. In this paper, the quantum capacities of the attenuation channel for PSK and OOK are discussed.

1. INTRODUCTION

In the development of quantum information theory, the concept of channels has played an important role. In particular, an attenuation channel has received much attention in optical communication. Information is expressed by a quantum state, and we are interested in how much information carried by the state is correctly transmitted to a receiver. This amount of transmitted information from an input to an output through the quantum channel is expressed by the quantum mutual entropy introduced in Ohya (1983). Based on the quantum mutual entropy, the quantum capacity for a quantum channel was studied in Ohya *et al.* (n.d.), which is a tool measuring the ability for information transmission of a quantum channel.

In this paper, we compute the quantum capacity of an attenuation channel with two fixed modulations, OOK (On–Off–Keying) and PSK (Phase–Shift–Keying), under certain energy constraint and discuss the efficiency of information transmission of OOK and PSK.

2. QUANTUM CHANNEL, MUTUAL ENTROPY, AND CAPACITY

Let $\mathcal{H}_1$ (resp. $\mathcal{K}_1$) and $\mathcal{H}_2$ (resp. $\mathcal{K}_2$) be the Hilbert spaces of input (resp. noise) and output (resp. loss) systems, let $B(\mathcal{H}_j)$ [resp. $B(\mathcal{K}_j)$] be the set of

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0020-7748/98/0100-050 7$15.00/0

all bounded operators on $\mathcal{H}_j$ (resp. $\mathcal{K}_j$) and $\mathfrak{S}(\mathcal{H}_j)$ [resp. $\mathfrak{S}(\mathcal{K}_j)$] be the set of all density operators on $\mathcal{H}_j$ (resp. $\mathcal{K}_j$) ($j = 1, 2$). A quantum channel Λ^* is a mapping from $\mathfrak{S}(\mathcal{H}_1)$ to $\mathfrak{S}(\mathcal{H}_2)$. The attenuation channel Λ^* (Ohya, 1983) is defined by

$$\Lambda^*(\rho) = tr_{\mathcal{K}_2} V(\rho \otimes |0\rangle\langle 0|)V^*, \qquad \forall \rho \in \mathfrak{S}(\mathcal{H}_1)$$

where $|0\rangle\langle 0|$ is the vacuum noise state in $\mathfrak{S}(\mathcal{K}_1)$, $tr_{\mathcal{K}_2}$ is the partial trace with respect to $\mathcal{K}_2$, and V is the mapping from $\mathcal{H}_1 \otimes \mathcal{K}_1$ to $\mathcal{H}_2 \otimes \mathcal{K}_2$ given by

$$V(|n\rangle \otimes |0\rangle) = \sum_{j=0}^{n} \sqrt{\frac{n!}{j!(n-j)!}\, \eta^j (1-\eta)^{n-j}}\, |j\rangle \otimes |n-j\rangle$$

for any photon number state vector $|n\rangle \in \mathcal{H}_1$ with a transmission rate η.

The quantum mutual entropy was introduced in Ohya (1983) such that

$$I(\rho; \Lambda^*) = \sup\left\{\sum_n \lambda_n (tr\Lambda^* E_n(\log \Lambda^* E_n - \log \Lambda^*\rho)); \rho = \sum_n \lambda_n E_n\right\}$$

where the supremum is taken over all von Neumann–Schatten decompositions (Schatten, 1970) $\sum_n \lambda_n E_n$ of ρ. Using the quantum mutual entropy, the quantum capacity for a quantum channel Λ^* with respect to the subset $\mathcal{S}$ of the state space $\mathfrak{S}$ $(\mathcal{H}_1)$ was defined in Ohya *et al.* (1997) as

$$C_q^{\mathcal{S}}(\Lambda^*) = \sup\{I(\rho; \Lambda^*); \rho \in \mathcal{S}\}$$

3. NUMERICAL COMPUTATION OF QUANTUM CAPACITY

In order to compute the quantum capacity, we have a useful proposition.

Proposition 1. For any input state $\rho = \lambda|x\rangle\langle x| + 1(1-\lambda)|y\rangle\langle y|$ with nonorthogonal normal vectors $x, y \in \mathcal{H}_1$, the von Neumann–Schatten decomposition of ρ is uniquely written

$$\rho = \|\rho\| E_0^{x,y} + (1 - \|\rho\|)E_1^{x,y}$$

where $E_j^{x,y} = |e_j^{x,y}\rangle\langle e_j^{x,y}| \in \mathfrak{S}(\mathcal{H}_1)$ with $|e_j^{x,y}\rangle = a_j|x\rangle + b_j|y\rangle$ ($j = 0, 1$). The above $\|\rho\|$ is given by

$$\|\rho\| = \frac{1}{2}\left(1 + \sqrt{1 - 4\lambda(1-\lambda)(1 - |\langle x, y\rangle|^2)}\right)$$

and the constants a_j, b_j are determined by

$$|a_j|^2 = \frac{(\tau_j^{x,y})^2}{(\tau_j^{x,y})^2 + 2|\langle x, y\rangle|\tau_j^{x,y} + 1}$$

$$|b_j|^2 = \frac{1}{(\tau_j^{x,y})^2 + 2|\langle x, y\rangle|\tau_j^{x,y} + 1}$$

$$a_j\bar{b}_j = \bar{a}_j b_j = \frac{\tau_j^{x,y}}{(\tau_j^{x,y})^2 + 2|\langle x, y\rangle|\tau_j^{x,y} + 1}$$

$$\tau_1^{x,y} = -\frac{1 + |\langle x, y\rangle|\tau_0^{x,y}}{\tau_0^{x,y} + |\langle x, y\rangle|}$$

$$\tau_0^{x,y} = \frac{-(1 - 2\lambda) + \sqrt{1 - 4\lambda(1 - \lambda)(1 - |\langle x, y\rangle|^2))}}{2(1 - \lambda)|\langle x, y\rangle|}$$

From the above proposition and Lemma 3 in Ohya (1983), the quantum mutual entropy with respect to the input states ρ and the attenuation channel Λ^* is calculated as

$$I(\rho; \Lambda^*) = S(\Lambda^*\rho) - \|\rho\|S(\Lambda^* E_0^{x,y}) - (1 - \|\rho\|)S(\Lambda^* E_i^{x,y})$$

where $S(\Lambda^*\rho)$ is the von Neumann entropy (Ohya and Petz, 1993) of the output state $\Lambda^*\rho$.

Let $\mathcal{S}_{PSK}$ and $\mathcal{S}_{OOK}$ be the subsets of $\mathfrak{S}(\mathcal{H}_1)$ given by

$$\mathcal{S}_{\mathrm{PSK}} = \{\rho = \lambda|\theta\rangle\langle\theta| + (1 - \lambda)|-\theta\rangle\langle-\theta|;\ \lambda \in [0, 1],\ \theta \in \mathbf{C}\}$$

$$\mathcal{S}_{\mathrm{OOK}} = \{\rho = \lambda|0\rangle\langle 0| + (1 - \lambda)|\theta\rangle\langle\theta|;\ \lambda \in [0, 1],\ \theta \in \mathbf{C}\}$$

where $|\theta\rangle\ \langle\theta|$ and $|-\theta\rangle\ \langle-\theta|$ are coherent states in $\mathfrak{S}(\mathcal{H}_1)$. The quantum capacities of the attenuation channel Λ^* with respect to the above two sets are computed under an energy constraint $|\theta|^2 \leq t$ for any $t \geq 0$:

$$C_q^M(\Lambda^*)i = \sup\{I(\rho; \Lambda^*);\ \rho \in \mathcal{S}_M,\ |\theta|^2 \leq t\}$$

where M represents PSK or OOK. We obtain the following result.

Theorem 2. (1) For $\rho = \lambda|\theta\rangle\ \langle\theta| + (1 - \lambda)\ |-\theta\rangle\ \langle-\theta|$, $S(\Lambda^*\rho)$ (i.e., $|x\rangle = |\theta\rangle$, $|y\rangle = |-\theta\rangle$ in Proposition 1), $S\ (\Lambda^* E_0^{\theta,-\theta})$, and $S(\Lambda^* E_1^{\theta,-\theta})$ are

$$S(\Lambda^*\rho) = -\sum_{i=0}^{1} \nu_i \log \nu_i, \qquad S(\Lambda^* E_j^{\theta,-\theta}) = -\sum_{i=0}^{1} \tilde{\mu}_{ji}, \log \tilde{\mu}_{ji}$$

$$\nu_i = \frac{1}{2}\left(1 + (-1)\ \sqrt{1 - 4\lambda(1 - \lambda)([1 - \exp(-4|\theta_\eta|^2)]}\right)$$

$$\tilde{\mu}_{ji} = \frac{1}{2}\left(1 + (-1)^i\ \sqrt{1 - 4\mu_j(1 - \mu_j)(1 - |\xi_j|^2)}\right)$$

where $\theta_\eta = \sqrt{\eta}\theta$ and

$$\xi_j = \frac{(\tau_j^{\theta,-\theta})^2 - 1}{\sqrt{((\tau_j^{\theta,-\theta})^2 + 1)^2 - 4\exp(-4|\theta_\eta|^2)(\tau_j^{\theta,-\theta})^2}} \qquad (j = 0, 1)$$

$$\mu_j = \frac{1}{2}\left(1 + \frac{\exp(-2|\theta|^2)}{\exp(-2\eta|\theta|^2)}\right)\frac{(\tau_j^{\theta,-\theta})^2 + 2\exp(-2\eta|\theta_\eta|^2)\tau_j^{\theta,-\theta} + 1}{(\tau_j^{\theta,-\theta})^2 + \exp(-2|\theta|^2)\tau_j^{\theta,-\theta} + 1}$$

$$(j = 0, 1)$$

(2) For $\rho = \lambda|0\rangle\langle 0| + (1-\lambda)|\theta\rangle\langle\theta|$, $S(\Lambda^*\rho)$ (i.e., $|x\rangle = |0\rangle$, $|y\rangle = |\theta\rangle$ in Proposition 1), $S(\Lambda^* E_0^{0,\theta})$, and $S(\Lambda^* E_1^{0,\theta})$ are

$$S(\Lambda^*\rho) = -\sum_{i=0}^{1} \nu_i \log \nu_i, \qquad S(\Lambda^* E_j^{0,0}) = -\sum_{i=0}^{1} \tilde{\mu}_{ji}, \log \tilde{\mu}_{ji}$$

$$\nu_i = \frac{1}{2}\left(1 + (-1)^i\sqrt{1 - 4\lambda(1-\lambda)[1 - \exp(-|\theta_\eta|^2)]}\right)$$

$$\tilde{\mu}_{ji} = \frac{1}{2}\left(1 + (-1)^i\sqrt{1 - 4\mu_j(1-\mu_j)(1 - |\xi_j|^2)}\right)$$

where

$$\xi_j = \frac{(\tau_j^{0,\theta})^2 - 1}{\sqrt{((\tau_j^{0,\theta})^2 + 1)^2 - 4\exp(-|\theta_\eta|^2)(\tau_j^{0,\theta})^2}} \qquad (j = 0, 1)$$

$$\mu_j = \frac{1}{2}\left(1 + \frac{\exp(-\frac{1}{2}|\theta|^2)}{\exp(-\frac{1}{2}\eta|\theta|^2)}\right)\frac{(\tau_j^{0,\theta})^2 + 2\exp(-\frac{1}{2}\eta|\theta_\eta|^2)\tau_j^{0,\theta} + 1}{(\tau_j^{0,\theta})^2 + \exp(-\frac{1}{2}|\theta|^2)\tau_j^{0,\theta} + 1}$$

$$(j = 0, 1).$$

(3) For any $t \geq 0$, we have

$$C_q^{\mathrm{OOK}}(\Lambda^*)_i \leq C_q^{\mathrm{PSK}}(\Lambda^*)_t$$

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